Manifold classification from the descriptive viewpoint

Iian Smythe[1]
joint with Jeffrey Bergfalk[2]

March 13, 2026

[1]University of Winnipeg

[2]University of Barcelona

Two questions

Our work is framed by the following two questions:

1. *How can we analyze classification problems for **manifolds** using descriptive set theory?*
2. *What can we say about their complexity in specific cases?*

Answering the first requires parametrizing classes of manifolds as standard Borel spaces on which the natural notion of isomorphism is analytic. We aim to be as ***general*** and ***modular*** as possible.

Answering the second involves locating these isomorphism relations within the Borel reducibility hierarchy.

Our work presents a unified framework for Question 1 and some specific examples for Question 2.

The paper

Our paper is now available here:

- J. Bergfalk and I. B. Smythe. Manifold classification from the descriptive viewpoint. Preprint. arXiv:2512.24996. 2025.

It is *long* at 112 pages…

But I hope to at least touch on all of its major results in this talk.

Table of contents

A parameter space of manifolds

Refresher on manifolds

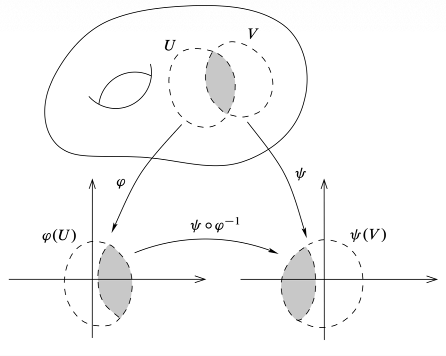
Recall that a **topological n-manifold** is a second countable Hausdorff space M which is "locally $\mathbb{R}^n$", i.e., every point has a neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

Any such M is covered by an atlas of **charts** (U, φ), where $U \subseteq M$ is open and φ a homeomorphism from U to an open subset of $\mathbb{R}^n$.

Refresher on manifolds

Other classes of manifolds are obtained by putting restrictions on the **transition maps** between overlapping charts (U, φ) and (V, ψ):

$$\psi \circ \varphi^{-1} : \varphi[U \cap V] \to \psi[U \cap V].$$



E.g., a manifold is **smooth** if all of the transition maps are smooth.

The pseudogroup framework

Definition

A **pseudogroup** $\mathcal{G}$ on a space X is a set of homeomorphisms between open subsets which cover X and is closed under:

- restriction
- composition
- inversion and
- arbitrary unions (when defining a homeomorphism).

Example

- For any X, the set Top of all homeomorphisms between open subsets of X is always the largest pseudogroup on X.
- For $X = \mathbb{R}^n$, the set C^∞ of all smooth homeomorphisms between open subsets of $\mathbb{R}^n$ is a subpseudogroup of Top.

$(\mathcal{G}, X)$-manifolds

Definition

Given a pseudogroup $\mathcal{G}$ on X, a space M is a $(\mathcal{G}, x)$**-manifold** if

- it is second countable and Hausdorff
- it is covered by a $(\mathcal{G}, X)$**-atlas** of **charts** (U, φ), each $U \subseteq M$ is open and φ is a homeomorphism of U with an open subset of X, and
- for any charts (U, φ), (V, ψ), the **transition map** $\psi \circ \varphi^{-1}$ lies in $\mathcal{G}$.

Example

- $(\mathrm{Top}, \mathbb{R}^n)$-manifolds are exactly the topological n-manifolds.
- $(C^\infty, \mathbb{R}^n)$-manifolds are exactly the smooth n-manifolds.

Note: We will focus exclusively on manifolds modeled on *locally compact* spaces X.

The Fell topology and Borel structure

When X is locally compact Polish, its set $\mathcal{F}(X)$ of closed subsets can be endowed with the **Fell topology**. This topology is Polish and induces the Effros Borel structure on $\mathcal{F}(X)$.

We consider the topology (which we also call the **Fell topology**) and Borel structure on the set $\mathcal{O}(X)$ of open subsets of X induced via complementation.

Note: Our requirement that X be locally compact ensures that $\cup$ is a Borel operation on $\mathcal{O}(X)$.

Borel pseudogroups

We next give a definability notion for pseudogroups which is compatible with the Fell topology on $\mathcal{O}(X)$.

Definition

If X is locally compact Polish, a pseudogroup $\mathcal{G}$ on X is **Borel** if it has a standard Borel structure making the following maps Borel:

1. domain projection: $\mathcal{G} \to \mathcal{O}(X) : \varphi \mapsto \mathrm{dom}(\varphi)$
2. range projection: $\mathcal{G} \to \mathcal{O}(X) : \varphi \mapsto \mathrm{ran}(\varphi)$
3. domain inclusion: $\mathcal{O}(X) \to \mathcal{G} : U \mapsto \mathrm{id}_U$
4. direct image: $\mathcal{O}(X) \times \mathcal{G} \to \mathcal{O}(X) : (U, \varphi) \mapsto \varphi[U]$
5. composition: $\mathcal{G} \times \mathcal{G} : (\varphi, \psi) \mapsto \varphi \circ \psi$
6. inversion: $\mathcal{G} \to \mathcal{G} : \varphi \mapsto \varphi^{-1}$.

Borel pseudogroups

Theorem (Bergfalk–S.)

For any locally compact Polish X, Top *can be endowed with the structure of a Borel pseudogroup.*

- The Borel structure is given by identifying each $\varphi \in \mathrm{Top}$ with $(U, V, (r_i)_{i\in\mathbb{N}}) \in \mathcal{O}(X) \times \mathcal{O}(X) \times X^{\mathbb{N}}$, where $U = \mathrm{dom}(\varphi)$, $V = \mathrm{ran}(\varphi)$, and $r_i = \varphi(q_i)$, for $(q_i)_{i\in\mathbb{N}}$ a fixed enumeration of some countable dense subset of X.
- When $X = \mathbb{R}^n$, C^∞ is a Borel subpseudogroup of Top.

The parameter space of $(\mathcal{G}, X)$-manifolds

When $\mathcal{G}$ is a pseudogroup on X, we define a set $\mathfrak{M}(\mathcal{G}, X)$ whose elements can be canonically associated to every $(\mathcal{G}, X)$-manifold.

Elements of $\mathfrak{M}(\mathcal{G}, X)$ are pairs $(\mathcal{U}, c)$ consisting of:

- a doubly-indexed sequence of open sets $\mathcal{U} = \langle U_{i,j} : i, j \in \mathbb{N} \rangle$, where $U_{i,j} \subseteq U_i := U_{i,i}$, and
- transition maps $c = \langle \varphi_{i,j} : U_{i,j} \to U_{j,i} : i, j \in \mathbb{N} \rangle$ in $\mathcal{G}$

which "glue together" U_i and U_j, subject to appropriate compatibility conditions.

The resulting $(\mathcal{G}, X)$-manifold is then $M_{(\mathcal{U},c)} = \coprod U_i / \sim$, where $x \sim y$ iff $\varphi_{i,j}(x) = y$ for some i, j. We further demand that $M_{(\mathcal{U},c)}$ is Hausdorff.

The natural notion of isomorphism for $(\mathcal{G}, X)$-manifolds then yields an equivalence relation $\cong_{\mathcal{G}}$ on $\mathfrak{M}(\mathcal{G}, X)$.

The parameter space of $(\mathcal{G}, X)$-manifolds

$\mathfrak{M}(\mathcal{G}, X)$ is then a subset of $\mathcal{O}(X)^{\mathbb{N}\times\mathbb{N}} \times \mathcal{G}^{\mathbb{N}\times\mathbb{N}}$ and forms **the parameter space of all $(\mathcal{G}, X)$-manifolds**.

Theorem (Bergfalk–S.)

If $\mathcal{G}$ is a Borel pseudogroup on a locally compact Polish X, then:

- $\mathfrak{M}(\mathcal{G}, X)$ *is a standard Borel space, and*
- $\cong_{\mathcal{G}}$ *is an analytic equivalence relation on* $\mathfrak{M}(\mathcal{G}, X)$.

As we vary $\mathcal{G}$ and X, we obtain parameter spaces for most commonly studied classes of finite-dimensional manifolds.

Compact and connected manifolds

Standard subclasses of $(\mathcal{G}, X)$-manifolds translate into subspaces of $\mathfrak{M}(\mathcal{G}, X)$. For example:

Lemma

If $\mathcal{G}$ is a Borel pseudogroup on a locally compact Polish X, then the subset $\mathfrak{K}(\mathcal{G}, X)$ of $\mathfrak{M}(\mathcal{G}, X)$, consisting of all $(\mathcal{U}, c)$ such that $M_{(\mathcal{U},c)}$ is ***compact****, is Borel.*

Lemma

If $\mathcal{G}$ is a Borel pseudogroup on a locally compact and locally path-connected Polish X, then then the subset $\mathfrak{C}(\mathcal{G}, X)$ of $\mathfrak{M}(\mathcal{G}, X)$, consisting of all $(\mathcal{U}, c)$ such that $M_{(\mathcal{U},c)}$ is ***connected****, is Borel.*

Compact manifolds

Our first Borel complexity calculation is a translation of the following result into this framework:

Theorem (Cheeger–Kister, 1970)

For each $n \in \mathbb{N}$, there are only countably many compact topological n-manifolds up to homeomorphism.

Corollary

For each $n \in \mathbb{N}$, $\cong_{\mathrm{Top}}$ on $\mathfrak{K}(\mathrm{Top}, \mathbb{R}^n)$ is Borel reducible to $=_{\mathbb{N}}$.

Corollary

Any assignment of topological invariants to compact manifolds can be viewed as a Borel operation.

- Similar results hold for compact smooth manifolds.

The classification of surfaces

Surfaces

By a **surface**, we mean a connected topological 2-manifold, which can be represented as an element of $\mathfrak{C}(\mathsf{Top}, \mathbb{R}^2)$.

By the previous result, any complexity here must lie among the *noncompact* surfaces.

Theorem (Bergfalk–S.)

The homeomorphism relation $\cong_{\mathsf{Top}}$ for surfaces is complete for countable structures.

- Thus, $\cong_{\mathsf{Top}}$ has the same complexity as the isomorphism relation for all countable groups, graphs, rings, Boolean algebras, etc.
- The same result also holds for the diffeomorphism relation $\cong_{C^\infty}$ on smooth surfaces.

Surfaces

Our result relies on a classical theorem:

Theorem (Kerékjártó 1923, Richards 1963)

Surfaces X and Y are homeomorphic iff they have the same ***genus****,* ***orientability class****, and the triples of* ***spaces of ends*** $(e(X), e'(X), e''(X))$ *and* $(e(Y), e'(Y), e''(Y))$ *are equivalent.*

- $(e(X), e'(X), e''(X))$ is a $\supseteq$-nested triple of compact totally disconnected spaces. $e(X)$ is the space of **ends** of X, $e'(X)$ the **nonplanar ends**, and $e''(X)$ the **nonorientable ends**.
- Ends are ways to go "to ∞" along a manifold, generalizing that $\mathbb{R}^2$ has "one end" at ∞ and $\mathbb{R}$ has "two ends" at $\pm\infty$.
- Such triples are **equivalent** if there is a homeomorphism of the first space which restricts to a homeomorphism on the others.

Lower bound

The lower bound in complexity for surfaces is not too difficult, as noted in a 2011 MathOverflow post by Clinton Conley:

Proof sketch.

Let C be the usual middle-thirds Cantor set lying in the sphere $S = \mathbb{R}^2 \cup \{\infty\}$.

Given any closed subset $K \subseteq C$, consider the resulting surface $S \setminus K$. The set of ends of $S \setminus K$ can be identified with K, and the corresponding nested triple is $(K, \emptyset, \emptyset)$.

This mapping gives a Borel reduction of the homeomorphism relation on closed subsets of C to that of surfaces, and the former is complete for countable structures by Camerlo–Gao (2001). $\square$

Upper bound

The upper bound in complexity, namely that homeomorphism of surfaces *is* classifiable by countable structures, is more difficult.

The proof requires showing that the end space construction can be carried out in a Borel way, and translated to countable structures, via a Borel version of Stone duality (again, using work of Camerlo–Gao). This is relatively straightforward, with the right construction.

However, we also need to compute combinatorial invariants of surfaces such as **genus** ($g \in \mathbb{N} \cup \{\infty\}$), **orientability** (“yes” or “no”), and **orientability class** (how Möbius bands accumulate along ends) in a Borel way.

Computing these combinatorial invariants classically uses a **triangulation** of the surface, i.e., a simplicial complex whose geometric realization is homeomorphic to the surface.

And most proofs of the existence of triangulations, in turn, rely on the Jordan–Schoenflies Theorem, a generalization of the Jordan Curve Theorem.

Thus, we needed to give **Borel** forms of these classical results.

Borel Triangulations

Thomassen (1992) gave graph-theoretic proofs of both of the aforementioned results which we adapt to the Borel setting:

Theorem (Bergfalk-S., "Borel Jordan–Schoenflies Theorem")

There is a Borel map $H : \mathrm{Emb}(S^1, \mathbb{R}^2) \to \mathrm{Homeo}(\mathbb{R}^2)$ *such that* $H(\gamma)|_{S^1} = \gamma$ *for all* $\gamma \in \mathrm{Emb}(S^1, \mathbb{R}^2)$.

Theorem (Bergfalk-S., "Borel Triangulation Theorem")

There is a Borel map $T : \mathfrak{C}(\mathsf{Top}, \mathbb{R}^2) \to \mathcal{S}$ *such that for each* $(\mathcal{U}, c) \in \mathfrak{C}(\mathsf{Top}, \mathbb{R}^2)$, $T(\mathcal{U}, c)$ *is a simplicial complex whose geoemtric realization is homeomorphic to* $M_{(\mathcal{U},c)}$.

- $\mathrm{Emb}(S^1, \mathbb{R}^2)$ and $\mathrm{Homeo}(\mathbb{R}^2)$ are spaces of embeddings and homeomorphisms, with the compact-open topology.
- $\mathcal{S}$ is the space of 2-dimensional simplicial complexes, viewed as countable structures.

Lie groups and hyperbolic manifolds

Hyperbolic space

In addition to topological or smooth manifolds, our framework can also accommodated "geometric" manifolds with metric structure.

The **upper-half space** $\mathbb{H}^n$ is the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ endowed with the **hyperbolic metric**.

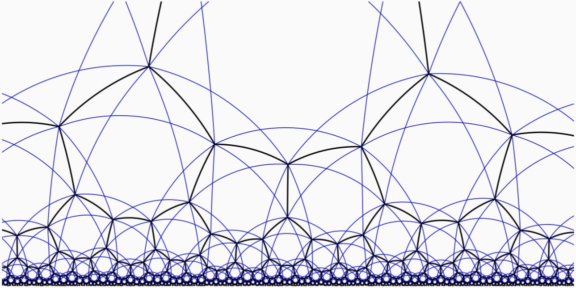


Figure 1: A tiling of $\mathbb{H}^2$ be hyperbolic heptagons

Hyperbolic manifolds

Let $\mathrm{Isom}(\mathbb{H}^n)$ be the group of isometries of $\mathbb{H}^n$ endowed with the compact-open topology. $\mathrm{Isom}(\mathbb{H}^n)$ can be given the structure of a Lie group, and in fact, a matrix group.

- The *orientation preserving* isometry groups $\mathrm{Isom}^+(\mathbb{H}^2)$ and $\mathrm{Isom}^+(\mathbb{H}^3)$ are familiarly identified with $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$.

$\mathrm{Isom}(\mathbb{H}^n)$ generates a Borel pseudogroup Isom on $\mathbb{H}^n$.

The resulting $(\mathsf{Isom},\mathbb{H}^n)$-manifolds are exactly the **hyperbolic *n*-manifolds**, which we consider up to isometry $\cong_{\mathsf{Isom}}$.

We focus on the class of (metrically) complete and connected hyperbolic n-manifolds, which in our framework correspond to a Borel subset $\mathfrak{C}_c(\mathsf{Isom},\mathbb{H}^n)$ of the parameter space $\mathfrak{M}(\mathsf{Isom},\mathbb{H}^n)$.

Subgroup conjugacy

If G is a locally compact Polish group, then its set $\mathcal{D}(G)$ of **discrete subgroups** forms a Polish space with the Fell topology, as does its subspace $\mathcal{D}_{\mathrm{tf}}(G)$ of **torsion-free discrete groups**.

G acts continuously on $\mathcal{D}(G)$ (and $\mathcal{D}_{\mathrm{tf}}(G)$) by **conjugation**:

$$g \cdot \Gamma = g\Gamma g^{-1}.$$

The resulting orbit equivalence relation is $E(G, \mathcal{D}(G))$ (or $E(G, \mathcal{D}_{\mathrm{tf}}(G))$).

Theorem (Kechris, 1992)

*If G is a locally compact Polish group acting in a Borel way on a standard Borel space X, the orbit equivalence relation E_X is **essentially countable**, i.e., bireducible with a countable Borel equivalence relation.*

Hyperbolic manifolds and subgroups of $\mathrm{Isom}(\mathbb{H}^n)$

There is a classical correspondence between complete connected hyperbolic manifolds up to isometry and torsion-free discrete subgroups of $G = \mathrm{Isom}(\mathbb{H}^n)$ up to conjugacy.

- One direction goes from subgroups Γ to manifolds $\mathbb{H}^n/\Gamma$.
- The other is via the "holonomy representation", taking a manifold M to a particular representation of $\pi_1(M)$ inside of G.

We show that this correspondence is Borel:

Theorem (Bergfalk–S.)

For all $n \geq 1$, the isometry relation $\cong_{\mathrm{Isom}}$ on $\mathfrak{C}_c(\mathrm{Isom}, \mathbb{H}^n)$ is Borel bireducible with the conjugacy relation $E(G, \mathcal{D}_{\mathrm{tf}}(G))$.

- Thus, $\cong_{\mathrm{Isom}}$ is essentially countable and Borel.

Complexity of conjugacy

The conjugacy relation has been well-studied in both descriptive set theory and ergodic theory. Let F_2 be the free group on 2 generators.

Theorem (Stuck–Zimmer, 1994)

$E(F_2, \mathcal{D}(F_2))$ does not Borel reduce to $=_{\mathbb{R}}$.

In fact:

Theorem (Thomas–Velickovic, 1999)

$E(F_2, \mathcal{D}(F_2))$ is a univseral countable Borel equivalence relation.

And more generally:

Theorem (Andretta–Camerlo–Hjorth, 2001)

If Γ is a countable discrete group containing F_2, then $E(\Gamma, \mathcal{D}_{\mathrm{tf}}(\Gamma))$ is a universal countable Borel equivalence relation.

Complexity of conjugacy

We prove a generalization of the Andretta–Camerlo–Hjorth theorem for a large class of Lie groups:

Theorem (Bergfalk–S.)

If G is a matrix group (i.e., a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$) which contains a discrete copy of F_2, then $E(G, \mathcal{D}_{\mathrm{tf}}(G))$ is an essentially universal countable Borel equivalence relation (i.e., is bireducible to a universal countable Borel equivalence relation).

Using standard Lie-theoretic techniques, this can extended to all noncompact semisimple Lie groups, sharpening another result of Stuck–Zimmer.

Back to hyperbolic manifolds

Returning to hyperbolic manifolds, since $G = \mathrm{Isom}(\mathbb{H}^n)$ is a matrix group which contains a discrete copy of F_2 whenever $n \geq 2$, we have:

Corollary (Bergfalk–S.)

For any $n \geq 2$, the isometry relation $\cong_{\mathsf{Isom}}$ on $\mathfrak{C}_c(\mathsf{Isom}, \mathbb{H}^n)$ is essentially universal countable.

Finitely generated subgroups and algebraically finite manifolds

Finitely generated subgroups

If Γ is countable, then it contains only countably many finitely generated subgroups. Consequently, the conjugacy relation on such subgroups trivially reduces to $=_{\mathbb{N}}$.

However, when we move to a locally compact group G, there may be uncountably many conjugacy classes of finitely generated (discrete) subgroups. It is this conjugacy relation we wish to study.

An instructive example comes from hyperbolic surfaces.

Closed surfaces and cocompact subgroups

Consider $G = \mathrm{Isom}(\mathbb{H}^2)$ and Γ a discrete torsion-free subgroup of G. We say that the surface $M = \mathbb{H}^n/\Gamma$ is **closed**, and Γ is **cocompact**, if M is compact. Let $\mathcal{D}_{cc}(G)$ be the set of cocompact subgroups of G.

The following facts are classical:

- Every cocompact Γ is finitely generated.
- For any hyperbolizable compact genus $g \geq 2$ surface S, the associated **moduli space** $\mathcal{M}(S)$ of all hyperbolic structures on S up to isometry is a quotient of the **Teichmuller space** $\mathrm{Teich}(S)$ of S by a properly discontinuous action of the **mapping class group** $\mathrm{MCG}(S)$. $\mathrm{Teich}(S)$ is a manifold of dimension $6g - 6$, and thus $\mathcal{M}(S)$ is an uncountable Polish space.

Corollary

Isometry of closed hyperbolic surfaces, and thus $E(G, \mathcal{D}_{\mathrm{cc}}(G))$, is Borel bireducible with $=_{\mathbb{R}}$.

Finiteness conditions

More generally, given $G = \mathrm{Isom}(\mathbb{H}^n)$ for $n \geq 2$, Γ a torsion-free discrete group of G, and $M = \mathbb{H}^n/\Gamma$, we say:

Definition

1. $M = \mathbb{H}^n/\Gamma$ is **closed**, and Γ is **cocompact**, if M is compact;
2. M is **finite volume**, and Γ is a **lattice**, if $\mathrm{vol}(M) < \infty$;
3. M and Γ are **geometrically finite** if Γ is finitely generated and the "convex core" of M of finite volume;
4. M is **algebraically finite** if Γ is finitely generated.

For any $n \geq 2$, we have that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$. When $n = 2$, we also have $3 \Leftrightarrow 4$.

We denote the corresponding (Borel) subsets of $\mathcal{D}_{\mathrm{tf}}(G)$ by $\mathcal{D}_{\mathrm{cc}}(G)$, $\mathcal{D}_{\mathrm{lat}}(G)$, $\mathcal{D}_{gf}(G)$, and $\mathcal{D}_{\mathrm{af}}(G)$, respectively.

Algebraic finiteness in dimension 2

When $n = 2$, the argument given for closed surfaces above also applies in the finite volume case, showing that $E(G, \mathcal{D}_{lat}(G))$ is bireducible with $=_\mathbb{R}$. More generally:

Theorem (Stuck–Zimmer, 1994)

*For **any** Lie group G, $E(G, \mathcal{D}_{\mathrm{lat}}(G))$ Borel reduces to $=_\mathbb{R}$.*

We extend this to all finitely generated torsion-free discrete groups in the case of $G = \mathrm{Isom}(\mathbb{H}^2)$:

Theorem (Bergfalk–S.)

Isometry of algebraically finite hyperbolic surfaces, and thus $E(G, \mathcal{D}_{\mathrm{af}}(G))$, is Borel bireducible with $=_\mathbb{R}$.

Algebraic finiteness in dimension 3

In contrast, when $n = 3$, geometric and algebraic finiteness no longer coincide, and the picture changes substantially:

Theorem (Bergfalk–S.)

For $G = \mathrm{Isom}(\mathbb{H}^3)$, the Borel complexity degrees of the finiteness classes of hyperbolic 3-manifolds M (and discrete torsion-free subgroups Γ of G) are:

- *for finite volume M (lattices Γ), it is $=_{\mathbb{N}}$;*
- *for geometrically finite M (Γ), it is $=_{\mathbb{R}}$;*
- *for algebraic finite M (finitely generated Γ), it is at least E_0.*

In particular, $E(G, \mathcal{D}_{\mathrm{af}}(G))$ does not Borel reduce to $=_{\mathbb{R}}$.

Algebraic finiteness in dimension 3

Where do these conclusions come from?

- "for finite volume M (lattices Γ), it is $=_{\mathbb{N}}$": This is essentially Mostow–Prasad rigidity; these manifolds are classified by their fundamental groups, of which there are only countably many.
- "for geometrically finite M (Γ), it is $=_{\mathbb{R}}$": This is the analog of the $n = 2$ case, and is related to the Ending Lamination Theorem of Brock, Canary, and Minsky (2010/2012) which classifies "marked" hyperbolic 3-manifolds.
- "for algebraic finite M (finitely generated Γ), it is at least E_0": This arises from a coarse embedding of the shift space $\{0, 1\}^{\mathbb{Z}}$ into the space of hyperbolic 3-manifolds of the form $S \times \mathbb{R}$, for closed surfaces S, produced by Abert, Bergeron, Biringer, et al (2020). In fact, isometry restricted to such manifolds is of complexity exactly E_0, by a result of Przytycki and Sabok (2021).

Problems for future study

Problems

This work raises many more interesting questions concerning manifold classification problems than it answers. Here a just a few:

Problem

What is the complexity of the homeomorphism relation on (connected) topological n-manifolds when $n \geq 3$?

- Iannella and Weinstein have recently announced a solution to this problem when $n = 3$ (it is complete for countable structures), but other cases remain wide open.

Problems

As n goes from 1 to 3, the complexity of the homeomorphism relation weakly increases. Does that remain true for all n?

Problem

Does the map which sends a positive integer n to the homeomorphism relation on (connected) topological n-manifolds weakly increase with respect to $\leq_B$?

- Note that the obvious strategies, such as taking a product with $\mathbb{R}$, do not give reductions from n-dimensions to $n+1$-dimensions.
- While a positive answer here may seem intuitive, note that precisely the opposite happens when we go from $n = 2$ to $n = 3$ in the case of closed hyperbolic surfaces up to isometry.
- These questions also apply to the smooth category.

Problems

In dimensions $n \leq 3$, the smooth and topological categories essentially coincide, as every topological n-manifold carries a unique smooth structure. In our setting, this translates to a Borel reduction of diffeomorphism to homeormorphism in these dimensions.

When $n = 4$, however, there are uncountably many distinct smooth structures on $\mathbb{R}^4$, so it is natural to ask:

Problem

What is the complexity of the diffeormorphism relation on smooth structures on $\mathbb{R}^4$?

- Gompf and Panagiotopolous have recently announced a lower bound of E_0 for this relation.

Problems

Finally, returning to hyperbolic manifolds, Lie groups, and their finitely generated subgroups:

Problem

For $n \geq 4$, what is the complexity of the conjugacy relation on finitely generated discrete torsion-free subgroups of $\mathrm{Isom}(\mathbb{H}^n)$? Equivalently, what is the complexity of the isometry relation on algebraically finite hyperbolic n-manifolds?

And more generally:

Problem

Does there exist a Lie group G such that the conjugacy relation on its finitely generated discrete subgroups is not essentially hyperfinite?

Thank you!